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Relational Thinking: Learning Arithmetic in order to Promote Algebraic Thinking

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Trends in the curriculum reform propose that algebra should be taught throughout the grades, starting in elementary school. The aim should be to decrease the discontinuity between the arithmetic in elementary school and the algebra in upper grades. This study was conducted to investigate and characterise upper elementary school students understanding of the various types of number sentences and relational thinking characteristics. With the information gathered, we were then able to formulate a model for developing students' relational thinking skill. This skill, confirmed by several studies, is a significant foundation for elementary students in the transition from arithmetic to algebraic understanding.

Keywords**:** Algebraic thinking; Arithmetic; Relational thinking

Introduction

Algebra is an important component for students learning mathematics. However, in practice, it has been noted that most students frequently misconceive various concepts in school algebra. Therefore, over the past decade, there has been an increased international focus on reform efforts in mathematics education to address this issue (e.g., Carpenter, Levi, Franke, & Zeringue, 2005; Irwin & Britt, 2005; Knuth, Stephens, Mcneil, & Alibabi, 2006; National Council of Teachers of Mathematics [NCTM], 2000).

Reform recommendations have proposed that algebra should be taught throughout the grades, starting in early elementary school (NCTM, 1989; 2000). By viewing algebra as a strand in the curriculum from pre-kindergarten onwards, teachers can help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra in the middle grades and high school (NCTM. 2000, p.37). Carpenter et al. (2005), however, propose that the goal should not be to teach algebra to elementary students but should focus on reducing the current major discontinuity between the arithmetic learnt in elementary school and the algebra that students are expected to learn in upper grades.

An example of the current discontinuity between elementary school arithmetic and the algebra learnt in upper grades, relates to understanding the concept of the equal sign. Studies have found that most elementary school students understand the equal sign to be a symbol of the calculation, i.e. an equal sign is always followed by the answer. Only a few children in traditional elementary school classes recognise that the equal sign represents a relation, a symbol that expresses a relationship “the same as”.

In fact, understanding the concept of the equal sign is essential to algebraic understanding (Freiman & Lee, 2004). Recognizing that the equal sign expresses a relation is critical for learning algebra. The flexibility afforded by using the equal sign to express a relation can also provide students a concept for representing important ideas in learning arithmetic (Carpenter, Franke, & Levi, 2003).

Traditional arithmetic emphasised training students accurately in algorithms of computation, but in algebra students need to have skills in transforming successive equivalent algebraic expressions. Carpenter et al. (2005) proposed that arithmetic algorithms can be taught as procedures for solving algebraic equations, by focusing on relations rather than calculating the answer. While other studies (e.g., Carpenter et al, 2005; Falkner, Levi, & Carpenter, 1999; Irwin & Britt, 2005; Mcneil & Alibabi, 2005; Stephen, 2006) found that arithmetic concepts learnt in elementary school could be better aligned with the concepts of algebra needed to be learnt in upper grades.

One approach has been to promote students' ability to work with number sentences by developing their thinking strategies. Irwin and Britt (2005) argued that the method of compensation and equivalence that we use in transforming number sentences such as $99 + 78$ into $100 + 77$ may provide a foundation for algebraic reasoning. Other authors, including Carpenter and Franke (2001) and Stephens (2008) described the thinking underpinning this kind of strategy as relational thinking. Carpenter, Levi, Franke, and Zeringue (2005) argued that by promoting relational thinking rather than focusing on procedures for calculating answers, learning and instruction can be made more consistent with the kinds of knowledge that support the learning of algebra while at the same time supporting and enhancing the learning of arithmetic (p.53). In order to introduce elementary school students to early algebraic understanding, it is necessary for the teaching to shift its emphasis to the structure of number sentences.

The current Thai Mathematics curriculum was developed and implemented in 2003. Although algebra is one theme for developing students understanding of mathematics, there is little suggestion of what should be used in classroom activities. In the Thai curriculum there is a serious discontinuity between the arithmetic taught in elementary school and the algebra taught in upper grades. The elementary school arithmetic syllabus focuses upon computational performance, without attending to relations and fundamental properties of arithmetic operations. As a result, students generally experience difficulty when they begin to study algebra in upper grades.

This study was designed to investigate and characterize the degree to which elementary school students understand the concepts of number sentences and relational thinking characteristics. As confirmed in several earlier studies, these skills are the foundation on which elementary students rely when making the transition from arithmetic to algebraic understanding. Using the information gathered, we have proposed a model for developing relational thinking skills to assist students in making this transition.

Method

This study was exploratory in nature and used a qualitative case study design. The aim of the study was to:

1. Investigate and characterise the degree in which students in upper elementary school understand the concepts of number sentences and relational thinking characteristics, and
2. Design an instructional model to better develop these relational thinking skills and assist students in making the transition.

Participants

The study was conducted at a primary school in southern Thailand and involved a total sample of 176 upper elementary school students in the 10 - 12 year age group. Participating students came from a middle socio-economic home environment. The sample group was a convenience sample.

This study was divided into two phases. In the first phase we worked with 146 students to determine their level of understanding of the concepts of number sentences and relational thinking characteristics. The second phase involved working with a smaller sample group of 30 students, to design a model which can be used to better develop their relational thinking skills.

Data Collection and Analysis

Key questions to be investigated and analyzed from student responses during the first phase were:

1. How can we classify the students according to their understanding of number sentences and how many groups can we categorize?
2. What factors influenced each group of students in understanding and using the concept of relational thinking?
3. What types of number sentences allow us to distinguish between relational and computational thinkers?

Errors made and strategies utilised by students when solving number sentences reflect their notion of number sentences and relational thinking characteristics. Relational thinking enables student to successfully solve open number sentences such as $8+4=\_\_\_+5$ (Falkner et al., 1999). However, within this group of students further distinctions can be made. Hunter (2007) said that these distinctions are between students who use the computational form of thinking and those who use the relational form of thinking. Stephen (2008) proposes that number sentences involving two unknown numbers, such as 18 + (Box A) = 20 + (Box B) can encourage students to think relationally.

In order to answer the questions postulated above, the 146 students were given a pen and paper questionnaire which consisted of three types of number sentences.

1. Type I was the true/false number sentence. Students were asked to write briefly how they know it was true or false.
2. Type II was the number sentence with one missing number. Students were asked to write briefly how they found the value of the missing number.
3. Type III was the number sentence with more than one missing number. Students were also asked to write briefly how they found the value of each missing numbers.

Examples of items used in this study were as follow:

Type I number sentences

25+9-9 = 25,   78+64 = 64+78,   73-15 = 70-12,   3+7+48 = 10+48

Type II number sentences

25+70 = □+71,   □-20 = 54-19,   358+□-36 = 360

Type III number sentences

84-□ = 86-□,   65+□-32 = 60+□-32,   501+□ = 502+□+9

The researcher applied the results of previous studies, which investigated students' notion of number sentences, together with the SOLO model of Biggs and Collis (1991) to categorise the responses of the 146 students into four groups. Three key indicators were used to categorise the student responses:

1. Their notion of the equal sign,
2. Their understanding of the structure of number sentences, and
3. The forms of thinking that they used to solve number sentences.

After the student responses were categorised, 18 representative students from each group were selected for in-depth interviews. Three coders (the researcher and two elementary mathematics teachers) coded all 18 student interview protocols and responses from the paperwork. The data were independently coded, based on a double coding procedure described by Miles and Huberman (1994). The reliability among coders in this study was 91 %.

In the second phase, a model was formulated for developing relational thinking skills. This model consisted of three parts:

1. Content,
2. Nature of activities, and
3. Teaching methods.

The contents component of the model was designed based on information gathered from the first phase. Activities were designed based on the concept of generalisation activities described by Ellis (2007). Finally, the teaching method was created based on the constructivism theory, focusing on students to create their own knowledge. The model was then used to design 11 lesson plans for teaching to a sample group of 30 upper elementary school students.

The key question of this second phase was the instructional model, and how this could be best developed to help students improve their relational thinking skills.

Results and Discussion

Students' Understanding of Number Sentences and Relational Thinking

Applying the three indicators used to consider students responses, we were able to categorize the 146 students into four groups:

1. **Group 0** (Pre-structural): Students in this group could not analyse any structure in the number sentences. They could only reach conclusions by guessing, or they did not respond at all.
2. **Group 1** (Uni-structural): Students in this group could search for only one structure in the number sentences, as in the form *"problem = answer"*. They conceived the equal sign to mean the result of computation and used computational thinking to solve the number sentences. The notion of equality between the two sides of the equal sign was not understood (See Figure 1).

1. **Group C** (Computational thinking): Students in this group understood that the equal sign was a relational sign. However, they were only able to view the structure of number sentences in two separate parts, as in the form *"result on the left* = *result on the right."* So they could only come to conclusions by using the computational algorithm (see Figure 2).
2. **Group R** (Relational thinking): Students in this group understood correctly that the equal sign was a relational sign. The advantage for the students this group was that they were able to view the number sentences as a whole in the form *"expression on the left* = *expression on the right"*. So they could come to conclusions using relational thinking based on the principle of equivalence and compensation (see Figure 3).

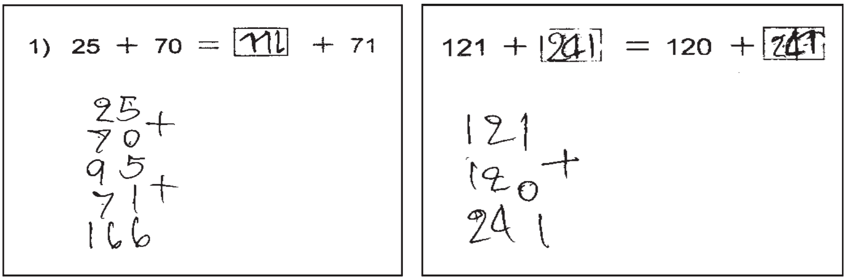

Figure 1. Student's misconception about the equal sign.

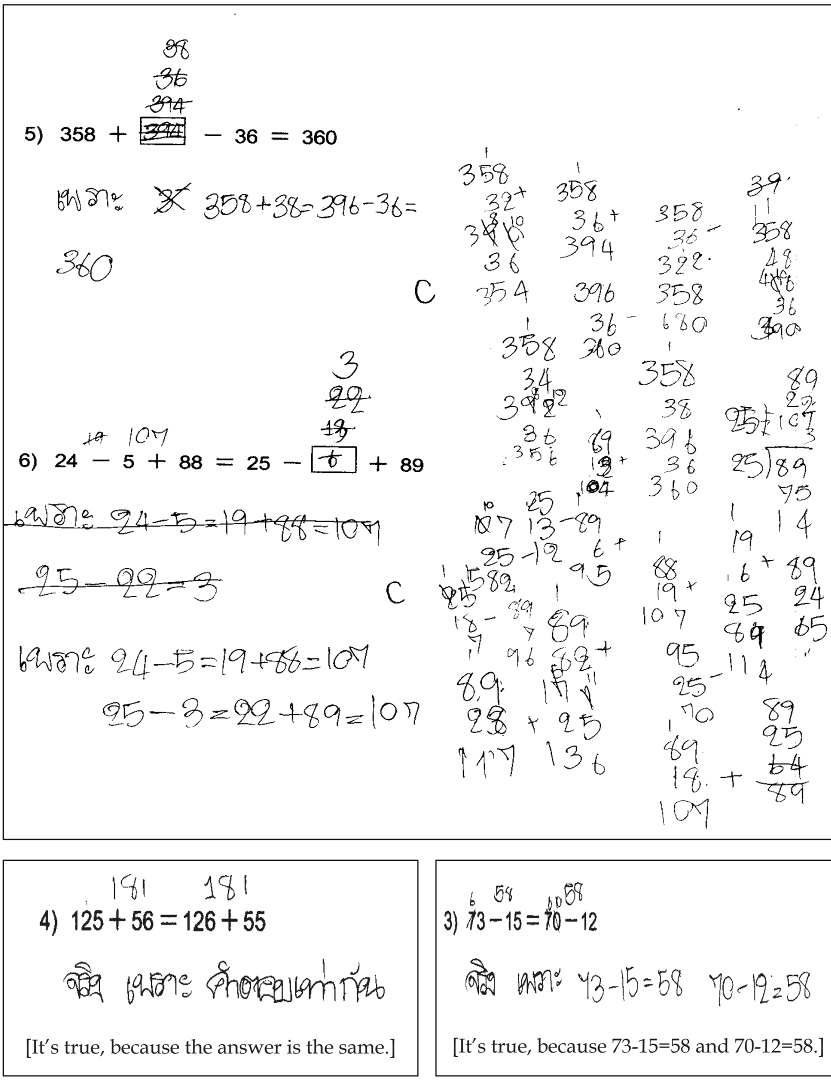

Figure 2. Students view the structure of number sentences in the form "result on the left = result on the right".

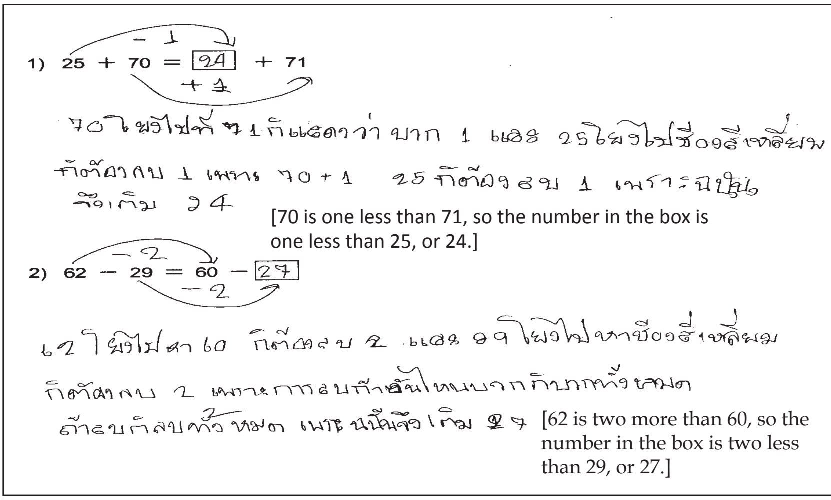

Figure 3. Students use relational thinking based on the principle of equivalence and compensation.

Obstacles to the Development of Relational Thinking Capability

Analysis of student responses from Group 0 and Group 1 indicated that the major problem for these students was a misconception about the meaning of the equal sign. Both groups regarded the equal sign as a symbol of the calculation. They believed that an equal sign is always followed by the answer. As a result, these students see the structure of the number sentences in the form of *"problem* = *answer"*. In order for students in these groups to improve their mathematical skills, activities should initially focus on developing a correct understanding of the meaning of the equal sign.

The major weakness of students in group C was their view of number sentences. They viewed the structure of number sentences as two distinct parts in the form *"result on the left* = *result on the right"*. This view resulted in members of this group using calculation methods to find the answer. Their development should focus on training them to see the sentence as a whole and find the relationship between each number.

For students in group R, errors most commonly occurred due to their lack of skill in using the equivalence and compensation principle and therefore an inability to expand the idea to the general case.

It can be concluded that the development of relational thinking skills requires a focus on four elements:

1. Develop a correct understanding about the meaning of the equal sign,
2. Train students to see the sentence as a whole,
3. Develop skills in using the equivalence and compensation principle, and
4. Expand the idea to the general case.

Relational Thinking and the Three Types of Number Sentences

Table 1 shows the number of times that the 146 students used relational thinking. The data shows clearly that the three types of number sentences encourage students to use relational thinking on different levels.

For Type I and Type II sentences, students used relational thinking only 29.45 and 39.50 percent respectively. However, the results were very different with the Type III sentences; these sentences encouraged students to use relational thinking as much as 61.19 percent. In other words, the form of the number sentence can stimulate students to use relational thinking, particularly in the case of number sentences similar to type III.

Table 1
The Number of Times that Students Use Relational Thinking

Grade / type of sentences	type I (6 items)	type II (6 items)	type III (6 items)	Total (18 items)
Grade 4(n=46)	36(276) 13.04 %	52(276) 18.84 %	120(276) 43.48 %	208(828) 25.12%
Grade 5(n=50)	75(300) 25.00 %	133(300) 44.33 %	197(300) 65.67 %	405(900) 45.00 %
Grade 6(n=50)	147(300) 49.00 %	161(300) 53.67 %	219(300) 73.00 %	527(900) 58.56 %
Total(n=146)	258(876) 29.45 %	346(876) 39.50 %	536 (876) 61.19 %	1,140 (2,628) 43.38 %

Model for Development of Relational Thinking Skills of Elementary School Students

From our analysis in Phase 1, we have created a model for developing relational thinking skills. This model consists of three parts (See Fig.4):

1. Content
2. Nature of activities
3. Teaching methods

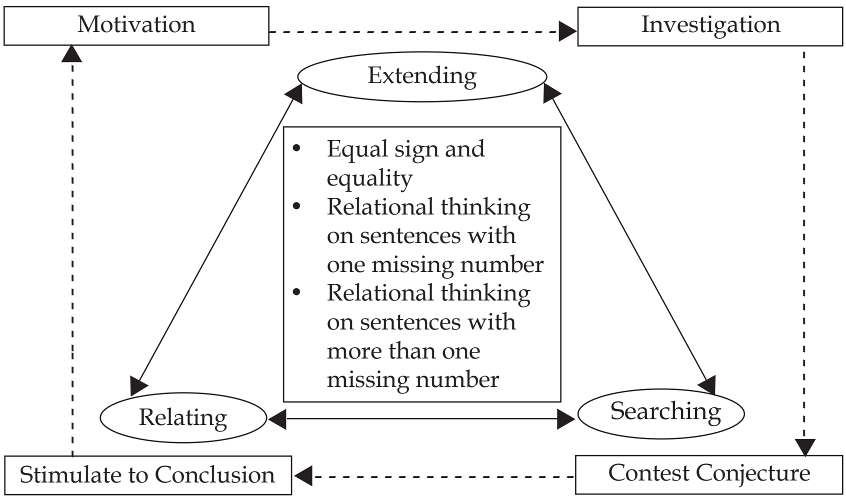

Figure 4. Model for developing relational thinking.

The core component of the model is content. The results of Phase 1 showed that the development of relational thinking skills involved four main components:

1. Developing a correct understanding of the equal sign,
2. Training students to see the sentence as a whole,
3. Developing skills in using the equivalence and compensation principle, and
4. Developing skills to expand the idea to the general case.

Since we found that type III number sentences encourage children to use relational thinking, we divide the contents into three parts:

1. Equal sign and equivalence,
2. Relational thinking on sentences with one missing number, and
3. Relational thinking on sentences with more than one missing number.

Each part emphasises the four main components.

The middle component of our model relates to activities having the following three characteristics:

1. ***Relating*** is an activity which aims to have students consider the similarity between number sentences, for example asking students to consider whether the sentences were true:

   $26+32=27+31$ and $552+45=553+44$

   And, asking them to judge whether or not the sentence

   $1,448+3,789=1449+3,788$ is true or false without resorting to calculation.

2. ***Searching*** is an activity which requires children to view the relation between given number sentences such as:

   $26+32=28+30,$

   $26+32=27+31,$

   $26+32=30+28$

   And asking them to find the missing number in the sentence

   $26+32=36+$ ___ without resorting to calculation

3. ***Extending*** is an activity which aims to develop children's ability to extend the relation from a specific to a general case, for example requiring children to complete a number sentence with two missing values such as $162+$ ___ $=160+$ ___ and then asking them to explain the relation between the values a and b for the number sentence $162+a=160+b$.

   The final component is the teaching method which consists of four steps

1. ***Motivation*** is the first step, where the teacher determines the problem in order to encourage students. In this step students should be encouraged to present their own ideas, and discuss in support of or argue against any other ideas presented by their classmates.

2. ***Investigation*** is the step where the teacher designs similar tasks to those used in the motivation step, but increases the number of items and asks students to investigate and make the mathematical conjectures by individual or group.
3. ***Contest Conjecture*** is the step where students communicate their mathematical conjectures to the class. In this step the teacher should encourage students to discuss the ideas that their classmates present.
4. ***Stimulate to Conclusion*** is the step in which teachers encourage students to reach a conclusion.

Changes in Relational Thinking Skills of 30 Target Students After Undertaking Trial Lessons

Based on the above model, we developed 11 lesson plans and taught them to a target group of 30 students over a four week trial period. The relational thinking skills of the students before and two weeks after the trial lesson period were then analysed and compared. The results are shown in Table 2.

Table 2
Relational Thinking Skill of The Students Before And After the Two-Week Trial Lesson Period

Grade / Group	0	1	C	R		Total
				NS	S	
Grade 4	1(0)	3(1)	3(1)	2(2)	0(5)	9
Grade 5	0(0)	2(0)	4(1)	3(2)	0(6)	9
Grade 6	0(0)	2(0)	6 (0)	2(5)	2(7)	12
Total	1(0)	7(1)	13(2)	7(9)	2(18)	30

a (b) number of students before (after) learning
NS non stable relational thinking student
S stable relational thinking student

From Table 2 it can be seen that prior to the trial learning period only 9 of the 30 students were using relational thinking (including 7 non stable relational thinkers). However, following the trial lessons, there was a marked improvement and 27 students were found to be using relational thinking techniques; 18 of these students were classified as stable relational thinkers.

Moreover, if considered individually, it was found that after the trial learning period most students had exhibited a degree of improvement in their ability to utilize relational thinking skills, even if they had not yet fully mastered the techniques. The most interesting point was that 18 of the 20 students in Group 1 and Group C had improved to a level where they could be reclassified into Group R. Furthermore, the 6 non stable relational thinking students had improved and were reclassified as stable relational thinking students (as shown in Table 3).

Table 3
Changes in Thinking Styles from Before and After Trial Lessons

Before Trial → After Trial Lessons	Number of students
$0 \to C$	1
$1 \to 1$	1
$1 \to C$	1
$1 \to R$	5
$C \to R(NS)$	5
$C \to R(S)$	8
$R(NS) \to R(NS)$	1
$R(NS) \to R(S)$	6
$R(S) \to R(S)$	2
Total	30

Conclusions and Implications

This study aimed initially to evaluate the relational thinking skills of a sample group of Thai upper elementary school students. A teaching model was then developed and a series of trial lessons were prepared and taught to a second group of students, with the aim of developing and improving their relational thinking skills.

A written questionnaire was used to present different types of number sentences to students and investigate their conceptions and strategies employed in solving these number sentence problems. Based on data from this phase of the study a teaching model was then developed and 11 lesson plans were prepared; focusing on four factors which were believed to contribute to developing the relational thinking skills of upper elementary students.

Results indicated that a major obstacle hindering the development of the students' relational thinking abilities was misconceiving the equal sign as a computational sign, rather than a sign expressing a relationship; i.e., that the answer comes next after the equal sign. This error is likely to have originated from elementary level textbooks that mostly present number sentences in the form *"problem = answer"*. Therefore, in order to introduce elementary school students to early algebraic thinking it is necessary for teaching to shift emphasis to the structure of equations. Recognising that the equal sign expresses a relation is critical for learning algebra, and the flexibility afforded by using the equal sign this way can also provide students a notation for representing important ideas in learning arithmetic (Kieran. 1989; Carpenter et al. 2003)

Furthermore it was found that developing student's skill in recognising number sentences as a whole, using the equivalence and compensation principle, and then being able to expand the concept to the general case are the foundations of relational thinking activity. The development of these skills was improved with exposure to the appropriate type of number sentences.

The data in this study show clearly that working with the three types of number sentences can encourage students to use and improve their ability to utilise relational thinking skills. The implications of this study suggest that teachers should give precedence to designing various number sentences appropriate with the three skills mentioned: relating, searching and extending activities. Even if some aspects of these three skills seem unreasonable for teaching at an elementary school level, if lessons and activities are carefully designed it is possible for most upper elementary school students to understand and benefit from the exposure to these concepts.

Both arithmetic and algebra are based on the same fundamental ideas. This means that we can manage the learning of arithmetic in elementary schools in order to promote the learning of algebra in upper grades. Relational thinking skill can support the development of algebraic reasoning while at the same time improving the learning and understanding of arithmetic.

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